1897MNRAS..57..133T

We have thus reduced the formulæ for the parentheses to the simple forms following :—

$$[\epsilon, \mu] = \frac{1}{2} a^{-\frac{1}{2}} \frac{da}{d\mu} \text{ for all which contain } \epsilon$$

$$[g, \mu] = \frac{\partial(g, h)}{\partial(g, \mu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad g$$

$$[\phi, \mu] = h \sin \phi \frac{\partial(\phi, \alpha)}{\partial(\phi, \mu)} \quad . \quad . \quad . \quad . \quad . \quad \phi$$

$$[\lambda, \mu] = \cos \phi \frac{\partial(\alpha, h)}{\partial(\lambda, \mu)} \text{ for all which contain only } a, e, \alpha$$

Perhaps the most instructive form is that which comprises *all* the fifteen parentheses in the general expression

$$[\lambda, \mu] = \tfrac{1}{2} a^{-\frac{1}{2}} \frac{\partial(\epsilon, a)}{\partial(\lambda, \mu)} + \frac{\partial(g, h)}{\partial(\lambda, \mu)} + \cos \phi \frac{\partial(\alpha, h)}{\partial(\lambda, \mu)} - h \sin \phi \frac{\partial(\alpha, \phi)}{\partial(\lambda, \mu)}.$$

We thus obtain by inspection

$$[\epsilon, a] = \frac{1}{2} a^{-\frac{1}{2}} \qquad [\epsilon, e] = 0 \qquad [\epsilon, \alpha] = 0 \qquad [\epsilon, \phi] = 0 \qquad [\epsilon, g] = 0$$

$$[g, a] = \frac{dh}{da} \qquad [g, e] = \frac{dh}{de} \qquad [g, \alpha] = 0 \qquad [g, \phi] = 0$$

$$[\phi, a] = 0 \qquad [\phi, e] = 0 \qquad [\phi, \alpha] = h \sin \phi$$

$$[e, a] = 0 \qquad [\alpha, a] = \cos \phi \frac{dh}{da} \qquad [\alpha, e] = \cos \phi \frac{dh}{de}$$

where

$$h = \sqrt{a(1 - e^2)}; \quad \frac{dh}{da} = \frac{1}{2} a^{-\frac{1}{2}} \sqrt{1 - e^2}; \quad \frac{dh}{de} = -\frac{a^{\frac{1}{2}} e}{\sqrt{1 - e^2}}$$

thus giving the well-known expressions in the Planetary Theory.

On Differential Refraction to Terms of Higher Orders than the First. By H. H. Turner, M.A., B.Sc., Savilian Professor.

1. For moderate zenith distances the differential refraction on, say, a photographic plate is expressed with sufficient accuracy by the first powers of the relative coordinates when the field is not too large ; but a plate may be exposed at a considerable zenith distance for the purpose of measuring parallax or some other reason, and again the field photographed may be of considerable extent. In such cases the first powers of the coordinates no longer afford a sufficiently accurate approximation, and we must take in

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

1897MNRAS..57..133T

higher powers. Investigations of this kind have been made by Rambaut (*Ast. Nach.* 3125), Kapteyn (*Bull. Com. Perm.* T. III.), and others; but I cannot but feel that the formulæ are greatly simplified by the use of rectangular coordinates such as I have advocated in previous papers (*M.N.* liv. p. 11, liv. p. 489, lv. p. 102, lv. p. 419, &c.), and in the present paper the differential refraction will be investigated by the use of such coordinates.

2. In *M.N.* liv. p. 18 I have given the formulæ for differential refraction as far as the first order. The expression there expanded is not, however, quite accurate, and, although correct to the second order, will not do beyond that order. If O be the centre of the plate, Ox, Oy the axes, and S, Z the projections on the plate of a star and the zenith respectively, then the effect of refraction is

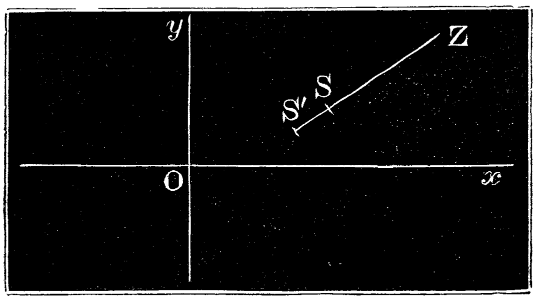

Fig. 1.

to displace S to S′ in the line S Z. So far this is rigidly accurate, since the great circle through the star and the zenith in which the refraction displacement takes place projects into the straight line S Z. But it is assumed in the paper above quoted that S S′ is proportional to tangent zenith distance, which is not quite accurate; in fact, S S′ is assumed equal to the arc of the sphere of which it is the projection; and this is only true to the second order of small quantities, *i.e.* when S is near O. We proceed to deduce the accurate formulæ.

3. Let (X, Y) be coordinates of Z; (x, y) of S; C the centre of the sphere; so that

$$x = \tan \mathrm{SCO} \,.\, \cos \mathrm{SO}x, \qquad y = \tan \mathrm{SCO} \,.\, \sin \mathrm{SO}x$$

$$\mathrm{X} = \tan \mathrm{ZCO} \,.\, \cos \mathrm{ZO}x, \qquad \mathrm{Y} = \tan \mathrm{ZCO} \,.\, \sin \mathrm{ZO}x.$$

Then the direction cosines of the line C S referred to rectangular axes at C are proportional to x, y, 1; and of C Z to X, Y, 1. Hence

$$\cos \mathrm{SCZ} = \frac{1 + x\mathrm{X} + y\mathrm{Y}}{\sqrt{(1 + x^2 + y^2)(1 + \mathrm{X}^2 + \mathrm{Y}^2)}},$$

whence we obtain

$$\tan \mathrm{SCZ} = \pm \frac{\sqrt{(x-\mathrm{X})^2 + (y-\mathrm{Y})^2 + (x\mathrm{Y} - y\mathrm{X})^2}}{1 + x\mathrm{X} + y\mathrm{Y}}.$$

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

1897MNRAS..57..133T

Now the coordinates of S′ may be written

$$x + t(x - \mathrm{X}), \qquad y + t(y - \mathrm{Y})$$

since it lies on the straight line S Z. Hence

$$\tan \mathrm{SCS}' = \pm \frac{t\sqrt{(x-\mathrm{X})^2 + (y-\mathrm{Y})^2 + (x\mathrm{Y} - y\mathrm{X})^2}}{1 + x^2 + y^2 + tx(x-\mathrm{X}) + ty(y-\mathrm{Y})}.$$

And

$$\tan \mathrm{SCS}' = \mu \tan \mathrm{SCZ},$$

where μ is the constant of refraction. Thus :

$$\pm t(1 + x\mathrm{X} + y\mathrm{Y}) = \mu\{1 + x^2 + y^2 + tx(x-\mathrm{X}) + ty(y-\mathrm{Y})\}.$$

Since the stars are displaced *towards* the zenith it is clear that the lower sign must be used if μ be taken positive. Hence

$$t = -\mu \,.\, \frac{1 + x^2 + y^2}{1 + x\mathrm{X} + y\mathrm{Y} + \mu\{x(x-\mathrm{X}) + y(y-\mathrm{Y})\}}$$

and the refractions in x and y are $t\ (x - \mathrm{X})$ and $t\ (y - \mathrm{Y})$ respectively.

In obtaining these results we have made no approximation beyond that of taking the refraction equal to its tangent, which is true to the second order of refraction.

4. We may now expand these expressions in powers of x and y. But before doing so we may reject the term in the denominator multiplied by μ. All the quantities being expressed in circular measure we have approximately

$$\mu = 1/3600 = \cdot 00028$$
$$\mu^2 = \cdot 00000008.$$

The effect of the denominator $\mathrm{P} + \mu\mathrm{Q}$ is to multiply the numerator by $\left(1 - \mu\dfrac{\mathrm{Q}}{\mathrm{P}}\right)\dfrac{1}{\mathrm{P}}$. Now x^2 and y^2 are small quantities ; if a star be 10° from the centre of the plate, which may be adopted as a superior limit for most photographs $x^2 + y^2$ is about 0·03. Thus the largest terms introduced into the coordinates of S′ by $\mu\mathrm{Q}$ may be taken equal to

$$+\mu^2\mathrm{Q}(\mathrm{X} - x)/\mathrm{P}^2.$$

Again it is only at great zenith distances where X and Y are large that such terms can be sensible. Hence we may neglect x and y in comparison with X and Y. Thus :

$$\mathrm{Q} = -(\mathrm{X}x + \mathrm{Y}y) \text{ and } \mathrm{P} = \mathrm{X}x + \mathrm{Y}y + 1.$$

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1897MNRAS..57..133T

For simplicity suppose $Y=0$, so that the axis of x is directed towards the zenith. Then the largest term of this kind under consideration is

$$\mu^2 X \,.\, Q/P^2 = \mu^2 X^2 \,.\, x/(1+Xx)^2.$$

Suppose the star 10° from the centre as before, so that $x = \tan 10°$. and let us put this term equal to 0″·025, or in circular measure $\frac{\pi}{180 \times 60 \times 60}$. Then

$$\frac{\pi}{180 \times 60 \times 2400} = \mu^2 X^2 \,.\, \frac{\pi}{18} \,.\Big/ \left(1 + \frac{1}{6} X^2\right)$$

or taking the square root

$$\frac{1}{1200} = \frac{1}{3600} X \Big/ \left(1 + \frac{1}{6} X\right)$$

or

$$3\left(1 + \frac{1}{6} X\right) \quad X.$$

Thus $X=6$, which corresponds to a zenith distance of 80° 32′. The term is clearly very small in all cases likely to occur in practice. Hence we may omit the term in the denominator multiplied by μ and write

$$t = -\mu \,.\, \frac{1 + x^2 + y^2}{1 + xX + yY}$$

with the refractions in x and y

$$\Delta x = (X - x)t, \qquad \Delta y = (Y - y)t$$

$$\begin{aligned} \therefore \frac{\Delta x}{\mu} = X & \\ & - x(1+X^2) \\ & + Xx^2(2+X^2) + Xy^2 \\ & - x^3(1+X^2)^2 - xy^2(1+X^2) \\ & + \&c. \end{aligned}$$

Since X must be greater than unity to make the higher orders sensible, the largest term of the second order is clearly X^3x^2, and of the third order X^4x^3.

In $\frac{1}{\mu}\Delta y$ there are no terms of the second and third orders comparable with these. The series will not be convergent if Xx is equal to or exceeds unity; but this would mean that the lower edge of the plate is below the horizon, which does not occur in practice. If, however, the plate is near the horizon, it is clear that the convergence is slow, and the expansion in series is not suitable; but the expressions are so simple that special cases can easily be considered individually. There is no need to go into wearisome detail here.

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1897MNRAS..57..133T

7. We may derive these expressions in a different manner, which has some interesting features. If we project upon the

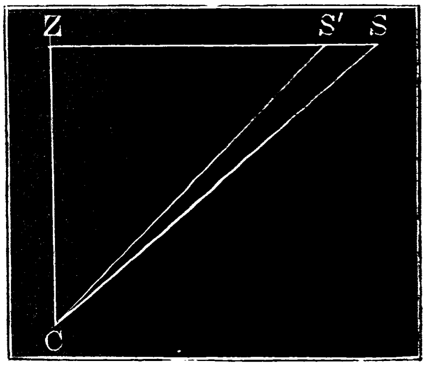

FIG. 2.

tangent-plane at the zenith, and S (fig. 2) be displaced to S′, C being the centre of the sphere and Z the zenith, we have

$$\theta = \angle \mathrm{SCS}' = \mu \tan \mathrm{ZD} = \mu \,.\, \mathrm{SZ} = \mu \,.\, r,$$

say; and

$$r + \Delta r = \mathrm{ZS}' = \tan(\mathrm{ZCS} - \theta)$$
$$= \frac{r - \mu r}{\mathrm{I} + \mu r^2}.$$

Thus

$$\Delta r = \frac{-\mu r(\mathrm{I} + r^2)}{\mathrm{I} + \mu r^2}.$$

Let (u, v) be the coordinates of S on the tangent-plane at Z. Then $r^2 = u^2 + v^2$ and

$$\mathrm{CS}^2 = \mathrm{I} + u^2 + v^2,$$

and

$$\Delta u = -\mathrm{SS}' \,.\, \frac{u}{\mathrm{ZS}} = -\mu u \,.\, \frac{\mathrm{I} + u^2 + v^2}{\mathrm{I} + \mu(u^2 + v^2)} = -\rho u, \text{ say}$$

$$\Delta v = -\mathrm{SS}' \,.\, \frac{v}{\mathrm{ZS}} = -\mu v \,.\, \frac{\mathrm{I} + u^2 + v^2}{\mathrm{I} + \mu(u^2 + v^2)} = -\rho v.$$

Now x and y are connected with u and v by relations of the form

$$x = \frac{au + bv + c}{\mathrm{I} - ku - lv}, \qquad y = \frac{du + ev + f}{\mathrm{I} - ku - lv}.$$

where $a, b, c \ldots$ are constants (see *Monthly Notices*, vol. liv. p. 11). Thus

$$x + \Delta x = \frac{au + bv + c - \rho(au + bv)}{\mathrm{I} - ku - lv + \rho(ku + lv)},$$

with a similar equation for $y + \Delta y$. Multiply up and use the value of x in terms of u and v to cancel equivalent terms.

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1897MNRAS..57..133T

Thus

$$\Delta x\{1 - ku - lv + \rho(ku + lv)\} = -\rho\{au + bv + x(ku + lv)\}$$
$$= \rho(c - x)$$

or on substituting

$$\rho = \frac{\mu(1 + u^2 + v^2)}{1 + \mu(u^2 + v^2)},$$

and simplifying we get

$$\Delta x = + \frac{\mu(c - x)(1 + u^2 + v^2)}{1 - ku - lv + \mu\{(u^2 + v^2) + ku + lv\}}.$$

8. There are relations between the coefficients a, b, c, &c., which may be obtained as follows. Let Z, O be the origins of the two systems of coordinates, S a star. Let the initial lines

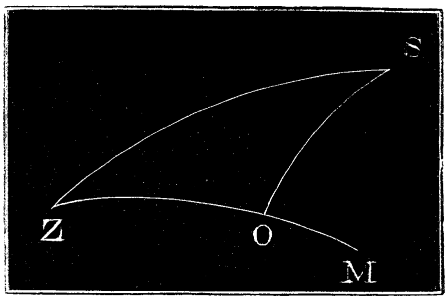

FIG. 3.

make angles a and β respectively with the line ZO; and let ZO $= \phi$. Then

$$u = \tan \text{ZS} \cos(\text{SZO} + \alpha), \qquad v = \tan \text{ZS} \sin(\text{SZO} + \alpha)$$
$$x = \tan \text{SO} \cos(\text{SOM} + \beta), \qquad v = \tan \text{SO} \sin(\text{SOM} + \beta).$$

Now

$$\sin \text{ZO} \sin \text{ZS} \cos \text{SZO} = \cos \text{SO} - \cos \text{SZ} \cos \text{ZO}$$
$$= \cos \text{SO}(1 - \cos^2 \text{ZO}) + \sin \text{SO} \sin \text{ZO} \cos \text{ZO} \cos \text{SOM}$$

Thus

$$\sin \text{ZS} \cos \text{SZO} = \cos \text{SO} \sin \text{ZO} + \cos \text{ZO} \sin \text{SO} \cos \text{SOM}$$

And

$$\sin \text{ZS} \sin \text{SZO} = \sin \text{SO} \sin \text{SOM}.$$

Also

$$\cos \text{ZS} = \cos \text{SO} \cos \text{ZO} - \sin \text{SO} \sin \text{ZO} \cos \text{SOM}.$$

Dividing the two previous equations by this, we get expressions for tan ZS cos SZO and tan ZS sin SZO; *i.e.* for $u \cos a + v \sin a$, and $-u \sin a + v \cos a$. The expressions are fractions the

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

1897MNRAS..57..133T

numerators and denominators of which may each be divided by cos SO : cos ZO ; and then $x \cos \beta + y \sin \beta$ and $-x \sin \beta + y \cos \beta$ may be substituted for tan SO cos SOM and tan SO sin SOM. We then obtain

$$u \cos \alpha + v \sin \alpha = \frac{\tan \phi + x \cos \beta + y \sin \beta}{1 - \tan \phi (x \cos \beta + y \sin \beta)}.$$

$$-u \sin \alpha + v \cos \alpha = \frac{\{-x \sin \beta + y \cos \beta\} \sec \phi}{1 - \tan \phi (x \cos \beta + y \sin \beta)}.$$

and similarly

$$x \cos \beta + y \sin \beta = \frac{u \cos \alpha + v \sin \alpha - \tan \phi}{1 + \tan \phi (u \cos \alpha + v \sin \alpha)}$$

$$-x \sin \beta + y \cos \beta = \frac{\{-u \sin \alpha + v \cos \alpha\} \sec \phi}{1 + \tan \phi (u \cos \alpha + v \sin \alpha)}.$$

9. Thus in the notation of § 7—viz.

$$x = \frac{au + bv + c}{1 - ku - lv}, \qquad y = \frac{du + ev + f}{1 - ku - lv}$$

we have

$$\begin{aligned}
a &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \sec \phi \\
b &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \sec \phi \\
c &= -\tan \phi \cos \beta \\
d &= \cos \alpha \sin \beta - \sin \alpha \cos \beta \sec \phi \\
e &= \sin \alpha \sin \beta + \cos \alpha \cos \beta \sec \phi \\
f &= -\tan \phi \sin \beta \\
k &= -\tan \phi \cos \alpha \\
l &= -\tan \phi \sin \alpha.
\end{aligned}$$

Further, we remark that the quantities (X, Y) being the values of x and y when $u=0$ and $v=0$ are

$$X = c = -\tan \phi \cos \beta, \qquad Y = f = -\tan \phi \sin \beta.$$

10. For the refraction we want the expressions $1 + u^2 + v^2$ and $1 - ku - lv$ in terms of x and y. Squaring the expressions for $u \cos \alpha + v \sin \alpha$ and $-u \sin \alpha + v \cos \alpha$, and adding we get $u^2 + v^2$, and to this we have to add unity. Perform the operation thus. Put for a moment

$$\xi = x \cos \beta + y \sin \beta, \qquad \eta = -x \sin \beta + y \cos \beta$$

and note that

$$\xi^2 + \eta^2 = x^2 + y^2$$

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1897MNRAS..57..133T

Then

$$1+(u\cos\alpha+v\sin\alpha)^2=1+\frac{\tan^2\phi+2\xi\tan\phi+\xi^2}{1-2\xi\tan\phi+\xi^2\tan^2\phi}$$

$$=\frac{(1+\xi^2)\sec^2\phi}{(1-\xi\tan\phi)^2}$$

$$(-u\sin\alpha+v\cos\alpha)^2=\frac{\eta^2\sec^2\phi}{(1-\xi\tan\phi)^2}$$

Thus

$$1+u^2+v^2=\frac{(1+x^2+y^2)\sec^2\phi}{(1-\xi\tan\phi)^2}$$

$$=\frac{(1+x^2+y^2)(1+X^2+Y^2)}{(1+xX+yY)^2}$$

from the expressions for X and Y given in the last paragraph.

Again

$$1-ku-lv=1+\tan\phi(u\cos\alpha+v\sin\alpha)$$

$$=1+\frac{\tan^2\phi+(x\cos\beta+y\sin\beta)\tan\phi}{1-\tan\phi(x\cos\beta+y\sin\beta)}$$

$$=\frac{1+X^2+Y^2}{1+xX+yY}$$

11\. Substituting these now in the expression of § 7—viz.

$$\Delta x=\frac{\mu(c-x)(1+u^2+v^2)}{1-ku-lv+\mu\{u^2+v^2+ku+lv\}}$$

and in the first instance neglecting the term multiplied by μ in the denominator we get

$$\Delta x=\mu\frac{(X-x)(1+x^2+y^2)}{1+Xx+Yy}$$

as in § 5. The neglected term is easily seen to be identical with that in § 3 and § 4 on remarking that the coefficient of μ^2 is

$$u^2+v^2+ku+lv\equiv(1+u^2+v^2)-(1-ku-lv)$$

$$=\frac{(1+X^2+Y^2)\{x(x-X)+y(y-Y)\}}{(1+xX+yY)^2}.$$

12\. The same algebraical expression applies when instead of the mean coordinates of the star we use the apparent coordinates, modifying the value of μ accordingly.

University Observatory, Oxford:
1897 *January* 7.

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System

1897MNRAS..57..133T

The Heliographic Coordinates of Sun-spots and Faculæ on the Stonyhurst Drawings. By the Rev. A. L. Cortie, S.J.

The photographic method of registering the positions of sun-spots and faculæ, and their micrometrical measurement from the photographs as practised at Greenwich, is doubtless the most perfect of those in use. Taking the positions as reduced, and published in the annual volumes of "Spectroscopic and Photographic Results" as standard, it will be useful to compare with them some positions which have been obtained from the Stonyhurst drawings by the use of a set of orthographic projections of the parallels of latitude and meridians of longitude, and to explain the method by which these positions have been obtained. For the process of measurement by means of projections is most simple and expeditious, and, moreover, is susceptible of considerable accuracy. The drawings themselves are made by projecting the image of the Sun on to a sketching-board, the diameter of the image being $10\frac{1}{2}$ inches. It is only necessary to rule one horizontal diameter of the circle prepared to receive the solar image in order to adjust the drawing to the apparent north point. This is effected by causing a spot to run truly along this diameter by means of the slow-motion rods of the telescope. It is better to choose a small spot for this purpose.

As the heliographic latitude of the centre of the Sun's disc varies between $\pm 7^\circ\ 15'$, eight orthographic projections of the meridians and parallels, corresponding to the fifteen positions of the pole ± 0 to $\pm 7^\circ$, were drawn with great care and exactness by Mr. William McKeon. They were then traced on glazed linen, so as to permit of their being placed over a drawing for measurement, all the details on the drawings being easily seen through the projection. Moreover they are very durable, and, as experience has proved, not liable to shrink, thus being an excellent substitute for costly projections ruled on glass.

Each of the projections has, moreover, a scale of degrees from 0 to $\pm 30^\circ$, ruled along its limbs north and south of the projection of the Sun's equator, to enable the disc to be adjusted for the varying position of the angle of inclination of the Sun's polar axis, this variation being within the limits $\pm 26^\circ{\cdot}5$ from the N. point. With these discs the "Ephemeris for Physical Observations of the Sun," published yearly in the *Companion to the Observatory*, must be used, values of P, D, and L being easily interpolated for the day and time of observation.

The use of the projections will be best illustrated by an example. To find the heliographic coordinates of a sun-spot drawn in the S.F. quadrant of the Sun at $9^h\ 37^m$ G.M.T. on 1893 Aug. 9. For this date and time $P = +14^\circ{\cdot}33$, $D = +6^\circ{\cdot}46$, $L = 266^\circ{\cdot}56$. Measure the apparent latitude and longitude by means of the projection for $D = \pm 6^\circ$, adjusting it with reference to the horizontal diameter ruled on the picture to the reading

M

Downloaded from http://mnras.oxfordjournals.org/ at Robertson Library University of prince Edward Island on July 22, 2015

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System